Existence of Nowhere Differentiable Boundaries in a Realistic Map

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A generic 3-dimensional diffeomorphic map, with constant Jacobian determinant, is proposed and looked at numerically. It contains a lower-dimensional basin boundary along which a chaotic motion takes place. This boundary is nowhere differentiable in one direction. Therefore, nowhere differentiable limit sets exist generically in nature.

If the world is a differential equation, as Boltzmann thought, the problem of the origin of the "nowhere differentiable nature of reality" (cf. Mandelbrot [1]) poses itself [2]. To our knowledge, only two differentiable maps exist in the literature which produce nowhere differentiable behavior, the 3-D diffeomorphisms indicated by Grebogi et al. [3] and by Kaneko [4]. The first map [3] which contains both a nowherin differentiable basin boundary and a nowhere differentiable attractor is, however, nongeneric. That is, it contains a 2-variable subystem which is Hamiltonian as a forcing subsystem, namely, Arnold's cat map. The second map [4] possesses a nowhere differentiable torus at exactly one point in parameter space.

We consider the following generic 3-D map:

$$\begin{aligned} x_{n+1} &= x_n^2 + a\,y_n\,, \\ y_{n+1} &= b\,y_n(1-y_n) + c\,z_n\,, \\ z_{n+1} &= x_n \end{aligned} \tag{1}$$

with $x_n, y_n, z_n, a, b, c \in \mathbb{R}$, $n = 1, 2, \ldots, N$. Its Jacobian determinant is equal to ac. It represents a C^∞ diffeomorphism, its explicit inverse exists and is differentiable, too.

The philosophy behind (1) is very simple. The first line describes a bistable subsystem. It possesses an unstable steady state at $x=1$ if $a=0$. At $x=0$ and $x=\infty$, respectively, there are point attractors. The second line gives a chaos-generating logistic-type map. In the case of $a \neq 0$, it "forces" the bistable system. The third line, as well as the linear coupling term in the second line, only serve to make the overall system diffeomorphic and, hence, realistic.

We focus on the basin boundary formed by the repelling fixed point of the first line near $x=1$. If the forcing parameter a is equal to zero, a straight line is obtained in the (x, y) plane of initial conditions. All values of x under iteration exceeding unity explode towards infinity, while all values smaller than unity converge towards the fixed point at $x=0$. This criterion ($x_n > 1$ or $x_n < 0.1$, respectively) has been used to "color" the initial conditions according to their fate.

If the forcing is switched on ($a > 0$), one typically obtains a picture like that shown in Figure 1. We report here only on the case where an attractor in the finite survives. If a becomes too large, almost all initial conditions explode. The borderline still resembles the "straight line" for $a=0$.

If one magnifies this boundary in the x-direction, as has been done in Fig. 2, a "wavy" structure appears. Figure 3 shows a series of subsequent blow-ups. In each case, the – to the eye – "flattest" portion of the rolling hills of Fig. 2 (or the previous blow-up, respectively) is presented.

As to interpretation, the preceding numerical results suggest that the boundary seen in the (x, y) plane of initial conditions (with $z=0$) of the above figures may be *nowhere differentiable*. That is, a Weierstrass-function-like behavior appears to have been found in a 2-D cross-section through the present 3-D diffeomorphism. This interpretation is also defensible on theoretical grounds [5].

Indeed, the idea for the present map was derived from a rewriting of the complex logistic map which is well known to generate a nowhere differentiable boundary. When written in polar coordinates (radius r,

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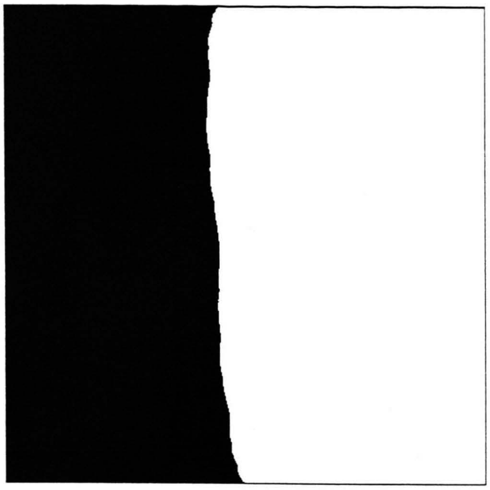

Fig. 1

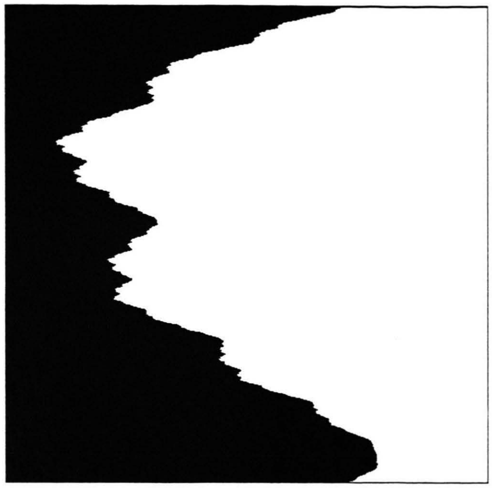

Fig. 3 (a)

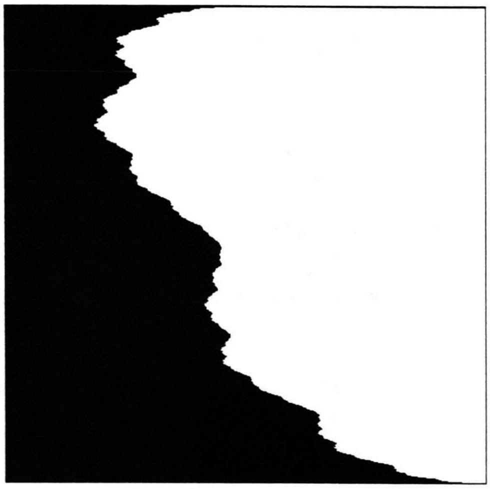

Fig. 2

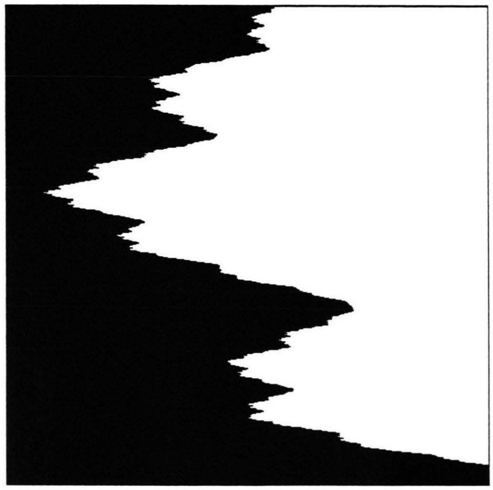

Fig. 3 (b)

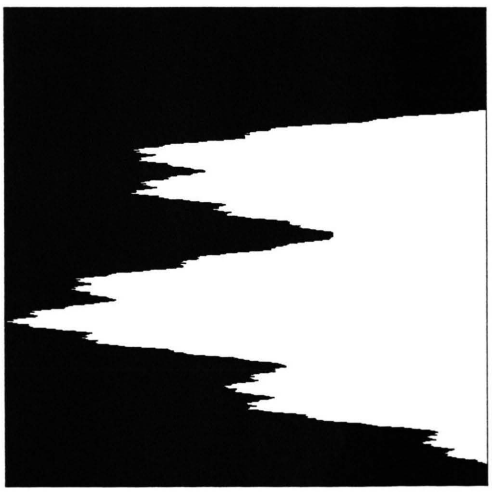

Fig. 3 (c)

Fig. 1. A 2-D cut ($z_0=0$) through a basin boundary formed in (1). Parameters: $a=0.1$, $b=3.99$, $c=0.001$. Initial conditions in x and y in a grid of 500×500 pixels. Coloring convention: nonexploding points ($x_n<0.1$) black, exploding points ($x_n>1$) white. Coordinates lower left corner: $x_{\text{low}}=0.5$, $y_{\text{low}}=0.0$; window size: $\Delta x=1.0$, $\Delta y=1.0$.

Fig. 2. A magnification, only along the x-axis, of the boundary shown in Figure 1. Coordinates lower left corner: $x_{\text{low}}=0.9$, $y_{\text{low}}=0.0$; window size: $\Delta x=0.1$, $\Delta y=1.0$.

Fig. 3. (a) Quadratic blow-up of the area indicated symbolically in Fig. 2. Coordinates lower left corner: $x_{\text{low}}=0.94454$, $y_{\text{low}}=0.47747$; window size: $\Delta x=10^{-6}$, $\Delta y=10^{-5}$. (b) The same with respect to (a). Coordinates lower left corner: $x_{\text{low}}=0.9445405545$, $y_{\text{low}}=0.4774684018$; window size: $\Delta x=10^{-10}$, $\Delta y=10^{-9}$. (c) The same with respect to (b). Coordinates lower left corner: $x_{\text{low}}=0.944540554552583$, $y_{\text{low}}=0.47746840273169$, window size: $\Delta x=10^{-14}$, $\Delta y=10^{-13}$.

angle (φ), this equation, too, consists of a single-variable switch (the radius with r^2) which is forced by a chaotic second variable (the angle with $2\varphi \bmod 2\pi$) [2, 6]. Recently, the same principle has been found to work numerically in a 4-variable generic ordinary differential equation of reaction-kinetic origin [6]. However, a mathematically hard argument can also be provided. Since the boundary contains a chaotic motion which persists forever, two arbitrarily close points on the boundary "decouple" completely in finite time. Therefore, the "mean forcing" experienced by the two points is bound to be finitely different.

The philosophy underlying the present map has already been seen, it appears, by Grebogi et al. [3] who, as mentioned, used an area-conserving 2-variable chaos-generating map to force a nonlinear switch (like the present first variable), without any back-coupling. The main difference is that the 2-variable chaos-generating map (y and z) used in the present case for the forcing represents a Hénon-type dissipative map [7]. Therefore, *no* nowhere differentiable attractor is formed near $x=0$, in contrast to the situation described by Grebogi et al. [3]. Indeed, this attractor ceases to exist, as soon as the forcing chaos is no longer strictly area-preserving. In contrast, as the present example is proof of, the basin boundary near $x=1$ turns out to be structurally stable, that is, is *generic.*

Modelling nature may lead to ordinary differential equations of low dimensions. Discrete mappings derive from ordinary differential equations as Poincaré return maps. They are invertible in time – in contrast to the complex analytic case. In this sense, the present diffeomorphism, (1), proves to be a generic example of nowhere differentiable structures in realistic systems.

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